Optimal Control of Two-level Quantum System with Energy Cost Functional

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1 Objectives of the work

- The steering problem for multi-level quantum system minimizing energy cost functional of the control will be discussed.
- The system is subjected to an external field with the minimum energy function.
- The problem will be illustrated in terms of the quantum spin up and spin down states of the Pauli two-level system.

2 Multi-level Quantum Control System

- We consider a forced (controlled) system represented by the state equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H_A|\psi(t)\rangle + i\hbar B|u(t)\rangle \tag{1}$$

where the system Hamiltonian operators H_A and B are taken to be matrices of dimensions $n \times n$ and $n \times m$ respectively.

- Applying the classical variational principle the state vector of the quantum dynamical system (1) can be represented in the form as

$$|\psi(t)\rangle = U(t-t_0)|\psi(t_0)\rangle + \int_{t_0}^{t} U(t-\tau)B|u(\tau)\rangle d\tau \tag{2}$$

where U is the unitary matrix operator corresponding to the Hamiltonian H_A.

- We assume that the eigenvalues $a_1, a_2, \ldots a_n$ of the system matrix operator H_A are distinct. Then the adjoint of the unitary operator $U(t)$ assumes the representation

$$U^+(t) = \sum_{r=1}^{n} e^{\frac{i}{\hbar} a_r t} P_r = \sum_{r=1}^{n} g_r(t) P_r \tag{3}$$

with $g_r(t) = e^{\frac{i}{\hbar} a_r t}, n = 1, 2, \ldots n.$

- Then the system state is given by taking $t_0 = 0$ with initial state $|\psi(0)\rangle$,

$$\begin{aligned} |\psi(t)\rangle &= U(t)\{|\psi(0)\rangle + \int_0^t \sum_{r=1}^{n} g_r(\tau) P_r B |u(\tau)\rangle d\tau\} \\ &= U(t)\{|\psi(0)\rangle + S_0 |W(t)\rangle\} \end{aligned} \tag{4}$$

- where

$$S_0 = [P_1 B, P_2 B, \ldots, P_n B] \tag{5}$$

- and

$$|W(t)\rangle = \begin{bmatrix} |w_1(t)\rangle \\ |w_2(t)\rangle \\ \vdots \\ |w_n(t)\rangle \end{bmatrix} \tag{6}$$

with $|w_r(t)\rangle = \int_0^t g_r(\tau)|u(\tau)\rangle d\tau$.

- **Definition.** The operator S_0 formulated in (5)is defined to be quantum controllability operator of the quantum control system (1).

3 Formulation of the Optimal Control Problem

- We have a quantum mechanical control system described in section-1 in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{C}^n)$ by the time evolution state vector as

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H_A|\psi(t)\rangle + i\hbar B|u(t)\rangle \tag{7}$$

.

- The optimal control problem is to find the controller $|u(t)\rangle \in \mathcal{L}^2(\mathbb{C}^m)$ which steers the initial state $|\psi(0)\rangle$ to the final state $|\psi(T)\rangle$ in $\mathbb{C}^n$.

- The controller $|u(t)\rangle \in \mathcal{L}^2(\mathbb{C}^m)$ minimizes the energy cost functional over the time interval $0 \le t \le T$ prescribed by

$$J(u) = \int_0^T < u^+(t)|Q|u(t)\rangle dt \tag{8}$$

  where Q is a positive definite self-adjoint operator in the respective Hilbert space of the controller $|u(t)\rangle$.

- Without loss of generality the operator Q in (8) of the cost functional may be taken to be unity operator I.

- Because, for a positive definite self-adjoint operator Q there exists a nonsingular operator P such that $Q = P^+P$.

- Now put $|v(t)\rangle = P|u(t)\rangle$.
- Then (8) becomes

$$J = \int_0^T \langle v^+(t)|v(t)\rangle dt = \||v\rangle\|^2 \tag{9}$$

the norm square of the vector $|v(t)\rangle$.

- And in this case, we have to replace $|u(t)\rangle$ by $|v(t)\rangle$ and B by BP^{-1} in (1).
- Hence, in general, we can take the cost functional (8) to be the norm functional as

$$J(u) = \||u(t)\rangle\|^2. \tag{10}$$

4 Outline of solution of the optimal control problem

- The quantum mechanical system has been formulated in an analogous way of a classical finite dimensional multi-variable control system.
- The optimal control of the multi-level quantum system defined in (10) is formulated in the Hilbert space $\mathcal{L}^2(\mathbb{C}^m)$ as the minimization problem of a norm functional of the space.
- In this section we shall see that optimal control of the quantum mechanical system exists uniquely in a finite dimensional linear manifold of the Hilbert space $\mathcal{L}^2(\mathbb{C}^m)$, the admissible space of the control vector.

- Let $M^2[0,T]$ be the linear manifold generated by the eigenfunctions $\{g_r(t), r = 1, 2, \ldots n\}$ of the Hermitian operator H_A of the dynamical system in the Hilbert space $\mathcal{L}^2(\mathbb{C}^n)$.
- We now construct an orthonormal basis of the linear manifold $M^2[0,T]$. Using Gram-Schmidt orthogonalization process, let us construct an orthonormal functions $\{\theta_i(t), i = 1, 2 \ldots n\}$ as

$$\begin{aligned} \beta_1 &= g_1 \\ \beta_r &= g_r - \sum_{s=1}^{r-1} \langle g_r, \theta_s \rangle \theta_s \end{aligned} \tag{11}$$

where $\theta_1 = \frac{\beta_1}{\|\beta_1\|}$ *and* $\theta_r = \frac{\beta_r}{\|\beta_r\|}$.

- We elaborate the above in a few more steps:

$$
\begin{aligned}
\beta_1 &= g_1 \\
\beta_2 &= g_2 - \langle g_2, \theta_1 \rangle \theta_1 \\
\beta_3 &= g_3 - \langle g_3, \theta_1 \rangle \theta_1 - \langle g_3, \theta_2 \rangle \theta_2 \\
\beta_4 &= g_4 - \langle g_4, \theta_1 \rangle \theta_1 - \langle g_4, \theta_2 \rangle \theta_2 - \langle g_4, \theta_3 \rangle \theta_3 \\
\ldots &\ldots \ldots \\
\beta_n &= g_n - \langle g_n, \theta_1 \rangle \theta_1 - \langle g_n, \theta_2 \rangle \theta_2 - \langle g_n, \theta_3 \rangle \theta_3 - \\
&\quad \ldots \langle g_n, \theta_{n-1} \rangle \theta_{n-1}
\end{aligned}
$$

with $\theta_1 = \frac{\beta_1}{||\beta_1||}$, $\theta_2 = \frac{\beta_2}{||\beta_2||}$, $\theta_3 = \frac{\beta_3}{||\beta_3||}$ and so on $\theta_n = \frac{\beta_n}{||\beta_n||}$.

We now write the eigenfunctions $g_r(t)$ as

$$
\begin{aligned}
g_1 &= \langle g_1, \theta_1 \rangle \theta_1 \\
g_2 &= \langle g_2, \theta_1 \rangle \theta_1 + \langle g_2, \theta_2 \rangle \theta_2 \\
g_3 &= \langle g_3, \theta_1 \rangle \theta_1 + \langle g_3, \theta_2 \rangle \theta_2 + \langle g_3, \theta_3 \rangle \theta_3 \\
g_4 &= \langle g_4, \theta_1 \rangle \theta_1 + \langle g_4, \theta_2 \rangle \theta_2 + \langle g_4, \theta_3 \rangle \theta_3 + \langle g_4, \theta_4 \rangle \theta_4 \qquad (12) \\
\ldots & \ldots \ldots \\
g_n &= \langle g_n, \theta_1 \rangle \theta_1 + \langle g_n, \theta_2 \rangle \theta_2 + \langle g_n, \theta_3 \rangle \theta_3 + \\
& + \langle g_n, \theta_4 \rangle \theta_4 + \ldots + \langle g_n, \theta_n \rangle \theta_n
\end{aligned}
$$

- In a compact form we have

$$\begin{aligned} g_r(t) &= \|\beta_r\|\theta_r(t) + \textstyle\sum_{s=1}^{r} \langle g_r, \theta_s \rangle \theta_s(t) = \\ &= \textstyle\sum_{s=1}^{r} \langle g_r, \theta_s \rangle \theta_s(t),\ r = 1, 2, \ldots n. \end{aligned} \tag{13}$$

- Using the representations of the functions $g_r(t)$, $r = 1, 2, \ldots n$ given in (13), the adjoint $U^+(t)$ operator defined in (3) can be represented in terms of the orthonormal functions $\theta_r(t), r = 1, 2, \ldots n$ as

$$U^+(t) = \sum_{r=1}^{n} A_r \theta_r(t) \tag{14}$$

where

$$A_r = \langle g_r, \theta_r \rangle P_r + \langle g_{r+1}, \theta_r \rangle P_{r+1} + \ldots + \langle g_n, \theta_r \rangle P_n \tag{15}$$

- With elaboration we have

$$
\begin{aligned}
& \sum_{r=1}^{n} g_r(t)P_r \\
= \quad & g_1(t)P_1 + g_2(t)P_2 + g_3(t)P_3 + g_4(t)P_4 + \ldots + g_n(t)P_n \\
= \quad & \langle g_1, \theta_1 \rangle \theta_1 P_1 \\
& + \langle g_2, \theta_1 \rangle \theta_1 P_2 + \langle g_2, \theta_2 \rangle \theta_2 P_2 \\
& + \langle g_3, \theta_1 \rangle \theta_1 P_3 + \langle g_3, \theta_2 \rangle \theta_2 P_3 + \langle g_3, \theta_3 \rangle \theta_3 P_3 \\
& + \langle g_4, \theta_1 \rangle \theta_1 P_4 + \langle g_4, \theta_2 \rangle \theta_2 P_4 + \langle g_4, \theta_3 \rangle \theta_3 P_4 \\
& + \langle g_4, \theta_4 \rangle \theta_4 P_4 \\
& \ldots\ldots\ldots\ldots \\
& + \langle g_n, \theta_1 \rangle \theta_1 P_n + \langle g_n, \theta_2 \rangle \theta_2 P_n + \langle g_n, \theta_3 \rangle \theta_3 P_n \\
& + \ldots \langle g_n, \theta_n \rangle \theta_n P_n \\
= \quad & A_1\theta_1(t) + A_2\theta_2(t) + A_3\theta_3(t) + A_4\theta_4(t) + \ldots + A_n\theta_n(t)
\end{aligned}
\tag{16}
$$

- where

$$\begin{aligned} A_1 &= \langle g_1, \theta_1 \rangle P_1 + \langle g_2, \theta_1 \rangle P_2 + \langle g_3, \theta_1 \rangle P_3 \\ &\quad + \langle g_4, \theta_1 \rangle P_4 + \ldots + \langle g_n, \theta_1 \rangle P_n \\ A_2 &= \langle g_2, \theta_2 \rangle P_2 + \langle g_3, \theta_2 \rangle P_3 + \langle g_4, \theta_2 \rangle P_4 \\ &\quad + \ldots + \langle g_n, \theta_2 \rangle P_n \end{aligned} \tag{17}$$

  and so on.

- Now,

$$\begin{aligned} |w_1(t)\rangle &= \int_0^t g_1(\tau)|u(\tau)\rangle d\tau \\ &= \int_0^t \langle g_1, \theta_1 \rangle \theta_1(\tau)|u(\tau)\rangle d\tau \\ &= \langle g_1, \theta_1 \rangle \int_0^t \theta_1(\tau)|u(\tau)\rangle d\tau \\ &= \langle g_1, \theta_1 \rangle I_m |v_1(t)\rangle \end{aligned}$$

- with $|v_1(t)\rangle = \int_0^t \theta_1(\tau)|u(\tau)\rangle d\tau$ where $|u(\tau)\rangle$ is a $m \times 1$ column vector.
- Again,

$$\begin{aligned}
|w_2(t)\rangle &= \int_0^t g_2(\tau)|u(\tau)\rangle d\tau \\
&= \int_0^t \{\langle g_2, \theta_1\rangle\theta_1 + \langle g_2, \theta_2\rangle\theta_2\}|u(\tau)\rangle d\tau \\
&= \langle g_2, \theta_1\rangle \int_0^t \theta_1(\tau)|u(\tau)\rangle d\tau \\
&\quad + \langle g_2, \theta_2\rangle \int_0^t \theta_2(\tau)|u(\tau)\rangle d\tau \\
&= \langle g_2, \theta_1\rangle I_m|v_1(t)\rangle + \langle g_2, \theta_2\rangle I_m|v_2(t)\rangle
\end{aligned}$$

- Similarly,

$$
\begin{aligned}
|w_3(t)\rangle &= \langle g_3, \theta_1\rangle I_m |v_1(t)\rangle + \langle g_3, \theta_2\rangle I_m |v_2(t)\rangle \\
&\quad + \langle g_3, \theta_3\rangle I_m |v_3(t)\rangle
\end{aligned}
$$

- and

$$
\begin{aligned}
&|w_4(t)\rangle \\
&= \langle g_4, \theta_1\rangle I_m |v_1(t)\rangle + \langle g_4, \theta_2\rangle I_m |v_2(t)\rangle \\
&\quad + \langle g_4, \theta_3\rangle I_m |v_3(t)\rangle + \langle g_4, \theta_4\rangle I_m |v_4(t)\rangle
\end{aligned}
$$

- and

$$
\begin{aligned}
&|w_n(t)\rangle \\
&= \langle g_n, \theta_1\rangle I_m |v_1(t)\rangle + \langle g_n, \theta_2\rangle I_m |v_2(t)\rangle \\
&\quad + \langle g_n, \theta_3\rangle I_m |v_3(t)\rangle + \ldots + \langle g_n, \theta_n\rangle I_m |v_n(t)\rangle.
\end{aligned}
$$

- Then the vector function $|W(t)\rangle$ in (6) is transformed into

$$|W(t)\rangle = \triangle |V(t)\rangle \tag{18}$$

- where

$$|V(t)\rangle = \begin{bmatrix} |v_1(t)\rangle \\ |v_2(t)\rangle \\ \vdots \\ |v_n(t)\rangle \end{bmatrix} \tag{19}$$

with $|v_r(t)\rangle = \int_0^t \theta_r(\tau)|u(\tau)\rangle d\tau$

and $\triangle$ is a non-singular lower triangular matrix

$$\begin{pmatrix} \langle g_1, \theta_1 \rangle I_m & 0 & 0 & \cdots & 0 \\ \langle g_2, \theta_1 \rangle I_m & \langle g_2, \theta_2 \rangle I_m & 0 & \cdots & 0 \\ \langle g_3, \theta_1 \rangle I_m & \langle g_2, \theta_2 \rangle I_m & \langle g_3, \theta_3 \rangle I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_n, \theta_1 \rangle I_m & \langle g_n, \theta_2 \rangle I_m & \langle g_n, \theta_3 \rangle I_m & \cdots & \langle g_n, \theta_n \rangle I_m \end{pmatrix}$$

- such that

$$\triangle\triangle^{+} = D = [D_{rs}] \tag{20}$$

with D_{rs} are the scalar matrices of order M given by

$$D_{rs} = diag\{\langle g_r, g_s \rangle, \ldots, \langle g_r, g_s \rangle\}$$

- Then using (12) the state vector function $|\psi(t)\rangle$ described by (4) of the dynaminal system (7) may also be represented as

$$|\psi(t)\rangle = U(t)\{|\psi(0)\rangle + S|V(t)\rangle\}, \tag{21}$$

- where

$$S = [A_1B, A_2B, \ldots A_nB] \tag{22}$$

with A_r's defined in (15) and

$$|V(t)\rangle = \begin{bmatrix} |v_1(t)\rangle \\ |v_2(t)\rangle \\ \vdots \\ |v_n(t)\rangle \end{bmatrix}, |v_r(t)\rangle = \int_0^t \theta_r(\tau)|u(\tau)\rangle d\tau \tag{23}$$

- Putting the values of A_r's in (22) from (15) we get the algebraic relation

$$S = S_0 \triangle \tag{24}$$

where S_0 is defined by (5).

- $\triangle$ is a nonsingular lower triangular matrix given by

$$\triangle = \begin{bmatrix} \triangle_{11} & 0 & 0 & \dots & 0 \\ \triangle_{21} & \triangle_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \triangle_{n1} & \triangle_{n2} & \triangle_{n3} & \dots & \triangle_{nn} \end{bmatrix} \tag{25}$$

- $\triangle_{ir}$'s are the scalar matrices of order m expressed as

$$\triangle_{ir} = diag\{\langle g_i, \theta_r\rangle, \langle g_i, \theta_r\rangle, \ldots \langle g_i, \theta_r\rangle, \; i \geq r \tag{26}$$

  and $\triangle_{ir} = 0$, the null matrix of order m, for $i < r$.

- The basic problem of the dynamical system (7) is to steer the initial state $|\psi(0)\rangle$ to the final state $|\psi(T)\rangle$ over time interval $[0, T]$ where the norm of the controller $|u(t)\rangle$ over the time interval is minimized.

- Putting $t = T$ in (20) the state function is represented as

$$|\psi(T)\rangle = U(T)\{|\psi(0)\rangle + S|V(T)\rangle\}, \tag{27}$$

- and hence we obtain the algebraic system

$$S|V\rangle = |Y\rangle \tag{28}$$

- with

$$|Y\rangle = U^{+}(T)|\psi(T)\rangle - |\psi(0)\rangle \tag{29}$$

- The solution of the algebraic problem for classical control system of solving (27) with minimum norm $\|V\|$ for which $\min \|V\| = \min \|u\|$, the norm of the controller has been formulated in section-3 of [3].

- We now enunciate the main result in the following theorem:

  Theorem - 1 Given the initial state $|\psi(0)\rangle$ and the target state $|\psi(T)\rangle$ in the space $\mathbb{C}^n$, if the rank of the controllability operator S is n, then there exists a unique optimal vector $|\hat{V}\rangle$ of the algebraic system given by equation (27) with minimum norm and the optimal solution is given by

$$|\hat{V}\rangle = S^{+}(SS^{+})^{-1}|Y\rangle. \tag{30}$$

- The optimal control $|\hat{u}(t)\rangle$ of the dynamical system (7) minimizing the energy cost functional(10) is expressed in the form [3] given by

$$|\hat{u}(t)\rangle = \sum_{r=1}^{n} \theta_r(t)|\hat{v}_r(T)\rangle \tag{31}$$

- where the vectors $|\hat{v}_r(T)\rangle \in \mathbb{C}^m$ are given in (22) with

$$|\hat{V}(t)\rangle = \begin{bmatrix} |v_1\hat{(t)}\rangle \\ |v_2\hat{(t)}\rangle \\ \vdots \\ |v_n\hat{(t)}\rangle \end{bmatrix} \tag{32}$$

5 Synthesis of the minimum energy control

- So far we have expressed the optimal control of the controllable system having minimum energy in terms of a sequence of orthonormal functions in a finite dimensional subspace of the Hilbert space.
- So there lies a difficulty in solving practical problems.
- The present section is concerned with the synthesis of the optimal control of the dynamical system in some explicit useful form.

- **Lemma-1**

  Let the $\triangle$ represents the lower triangular matrix defined in section 3 by equation (24). The product $\triangle\triangle^+$ is a nonsingular matrix which is expressed in the form

$$\triangle\triangle^+ = D = \begin{bmatrix} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & D_{22} & \dots & D_{2n} \\ \dots & \dots & \dots & \dots \\ D_{n1} & D_{n2} & \dots & D_{nn} \end{bmatrix}$$

  where D_{ij}'s are scalar submatrices of order m given by

$$D_{rs} = diag\{\langle g_r, g_s\rangle, \dots, \langle g_r, g_s\rangle\}.$$

- **Theorem -2.**

  If the rank of controllability matrix S_0 defined by (5) of the system (7) is n, then the optimal control $|\hat{u}(t)\rangle$ minimizing the energy cost functional (10) which transfer the state of the system from the initial state $|\psi(0)\rangle$ to the target state $|\psi(T)\rangle$ can be formulated as

$$|\hat{u}(t)\rangle = K(t)|Y\rangle,\ |Y\rangle = U^{-1}(T)|\psi(T)\rangle - |\psi(0)\rangle,\ 0 \leq t \leq T \tag{33}$$

  where $U^{+}(T)$ is given by (3) for $t = T$ and $K(t)$ is an $m \times n$ matrix function of $t (0 \leq t \leq T)$ written as

$$K(t) = F(t) S_0^{+} (S_0 D S_0^{+})^{-1}. \tag{34}$$

- $F(t) = [I_m(g_1), I_m(g_2), \ldots, I_m(g_n)]$,
- and $I_m(g_r)$ is a scalar matrix as

$$I_m(g_r) = \begin{bmatrix} g_r(t) & 0 & \ldots & 0 \\ 0 & g_r(t) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & g_r(t) \end{bmatrix}$$

6 Electron spin:Quantum two-state system

- The spin state of an electron is represented on $\mathbb{C}^2$ in the basis formed by the eigenstates of the spin operator

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{35}$$

- The control system is defined by

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = S_x |\psi(t)\rangle + i\hbar\alpha |u(t)\rangle \tag{36}$$

- The eigenvalues of S_x are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

- The eigenvectors are given by $|\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

$|\downarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- The projection operators, for $a_1 = \frac{\hbar}{2}$ and $a_2 = \frac{\hbar}{2}$, are

$P_{|\uparrow\rangle} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $P_{|\downarrow\rangle} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ respectively.

- The adjoint of unitary operator $U(t)$ is

$$U^{+}(t) = e^{\frac{i}{\hbar} S_x t} = e^{i a_1 t} P_{|\uparrow\rangle} + e^{i a_2 t} P_{|\downarrow\rangle}.$$

- The optical control of the system can then be synthesized using the explicit formula

$$|\hat{u}(t)\rangle = K(t)|Y\rangle \tag{37}$$

- with

$$K(t) = F(t)S_0^+(S_0 D S_0^+)^{-1} \tag{38}$$

- and

$$|Y\rangle = U^{-1}(T)|\psi(T)\rangle - |\psi(0)\rangle \tag{39}$$

- Now

$$S_0 = [P_1 B, P_2 B] = \alpha[P_1, P_2] \tag{40}$$

- Then

$$S_0^+ = \alpha \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{41}$$

- Also

$$F(t) = [g_1(t)I, g_2(t)I] \tag{42}$$

- Hence we have

$$\begin{aligned} & F(t)S_0^+ \\ = \quad & [g_1(t)I, g_2(t)I]\alpha \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \alpha[g_1(t)IP_1 + g_2(t)IP_2] \\ & = \alpha[g_1(t)P_1 + g_2(t)P_2] \end{aligned} \tag{43}$$

- Again

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{44}$$

- where

$$D_{11} = \begin{bmatrix} \langle g_1 g_1 \rangle & 0 \\ 0 & \langle g_1 g_1 \rangle \end{bmatrix} = \langle g_1(t) g_1(t) \rangle I \tag{45}$$

- and similarly $D_{12} = \langle g_1(t) g_2(t) \rangle I$, $D_{21} = \langle g_2(t) g_1(t) \rangle I$, $D_{22} = \langle g_2(t) g_2(t) \rangle I$.

- We then at once obtain

$$S_0 D S_0^+ = \alpha^2 (TP_1 + TP_2) = \alpha^2 T(P_1 + P_2) = \alpha^2 T. \tag{46}$$

- Now

$$\begin{aligned} K(t) &= F(t) S_0^+ (S_0 D S_0^+)^{-1} \\ &= \alpha[g_1(t)P_1 + g_2(t)P_2] \tfrac{1}{\alpha^2 T} \\ &= \tfrac{1}{\alpha T}[g_1(t)P_1 + g_2(t)P_2] \end{aligned} \tag{47}$$

- In special case, let us try to find $|\hat{u}(t)\rangle$ for which the system is transferred from $|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the state $|\psi(T)\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then

$$
\begin{aligned}
& |Y\rangle \\
= \quad & \left(e^{\frac{iT}{2}} P_1 + e^{-\frac{iT}{2}} P_1\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
= \quad & \left(\frac{e^{\frac{iT}{2}}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{e^{\frac{-iT}{2}}}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
= \quad & \left(\frac{e^{\frac{iT}{2}}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{e^{\frac{-iT}{2}}}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned} \tag{48}
$$

- Thus the optimal control of the two-level Pauli spin system minimizing the system is given by

$$\begin{aligned}
|\hat{u}(t)\rangle &= \frac{1}{\alpha T}[g_1(t)P_1 + g_2(t)P_2]|Y\rangle \\
&= \frac{1}{\alpha T}\left(\frac{e^{\frac{it}{2}}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{e^{\frac{-it}{2}}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \\
&\quad \left(\frac{e^{\frac{iT}{2}}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{e^{\frac{-iT}{2}}}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\left\{e^{\frac{i(t+T)}{2}}|\uparrow\rangle + e^{\frac{-i(t+T)}{2}}|\downarrow\rangle\right\} - |\psi(0)\rangle
\end{aligned} \tag{49}$$

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Thank You for your attention!